Enrollment No: ______________________ Exam Seat No:______________________

C. U. SHAH UNIVERSITY

Winter Examination-2021

Subject Name: Group Theory

Subject Code: 4SC05GRT1 **Branch: B.Sc. (Mathematics)**

Semester: 5 **Date: 17/12/2021** **Time: 11:00 To 02:00** **Marks: 70**

Instructions:

(1) Use of Programmable calculator & any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

Q-1 **Attempt the following questions:** **(14)**

a) Define: Group.
b) True or False: $(Z_n, +_n)$ is a group.
c) If $G = \{1, -1, i, -i\}$ is a group under multiplication then find $O(i)$.
d) Give an example of abelian group.
e) If G is a finite group and let $a \in G$, then $a^{O(G)} =$ ______.
f) If G is a group and $a \in G$, then
i). $O(a)|O(G)$ii). $O(G)|O(a)$
iii). $O(a) \nmid O(G)$ iv). None of these
g) State Fermat's theorem.
h) If $f = \begin{pmatrix} 1 & 2 & 34 & 5 \\ 2 & 3 & 41 & 5 \end{pmatrix} \in S_5$ then find f^{-1}.
i) Define: Normal Subgroup.
j) Permutation $f = (1\ \ 2\ \ 3)(4\ \ 5) \in S_6$ then find $O(f)$.
k) If $p \in N$ is a prime number, then $\phi(p) =$ _____.
l) A cycle of length 2 is called ________.
m) In any group G the order of the identity element e is ______.
n) In a group G, if $a^2 = e;\ \forall a \epsilon G$. Then
i). G is cyclic ii).G is Finite
iii).G is commutative iv).None of these.

Attempt any four questions from Q-2 to Q-8

Q-2 **Attempt all questions** **(14)**

a) Prove that $(G,\cdot)$ is a group where, $G = \{a + b\sqrt{3}\ |\ a, b \in Q, a^2 - 3b^2 \neq 0\}$ and $'\cdot'$ is the multiplication of real numbers. **05**
b) Let H_1 and H_2 be any two subgroups of a group G then show that $(H_1 \cap\ H_2)$ is also a subgroup of G. **05**
c) Let G be a group and let a be any element of G such that $O(a) = n$. then prove that **04**
(i) $O(a^m) \leq O(a), \forall m \in Z$
(ii) $O(a^{-1}) = O(a)$

Q-3		**Attempt all questions**	**(14)**
	a)	State and prove Euler's theorem.	**05**
	b)	Let H be a subgroup of G and $a, b \in G$ then show that $Ha = Hb \Leftrightarrow ab^{-1} \in H$.	**05**
	c)	Let G be a Group and let $a \in G$ then Prove that $N(a) = \{x \in G / xa = ax\}$ is a subgroup of G.	**04**
Q-4		**Attempt all questions**	**(14)**
	a)	State and Prove Lagrange's theorem.	**07**
	b)	Let H be a non-empty subset of a group G. Then prove that following are equivalent. (i) H is a subgroup of G. (ii) $\forall a, b \in H$ (a) $ab \in H$ and $(b) a^{-1} \in H$. (iii)$ab^{-1} \in H;\ \forall a, b \in H$	**07**
Q-5		**Attempt all questions**	**(14)**
	a)	Prove that H be a normal subgroup of a group G iff $aHa^{-1} \subset H$, for $\forall a \in G$.	**05**
	b)	Prove that composition of two disjoint cycle is commutative.	**05**
	c)	Let H be a subgroup of a group G, show that the relation $a \equiv b(mod\ H)$ is an equivalence relation in G.	**04**
Q-6		**Attempt all questions**	**(14)**
	a)	A subgroup H of a group$G$ is a normal subgroup iff $(Ha)(Hb) = H(ab);\ \forall a, b \in G$.	**07**
	b)	Using Euler's theorem, find the remainder when 3^{256} is devided by 14.	**04**
	c)	Let G be a group and H be a subgroup of G . Then prove that any Two right cosets of H in G are either identical or disjoint.	**03**
Q-7		**Attempt all questions**	**(14)**
	a)	Let H be a subgroup of G and $a, b \in G$ then show that$a \in H \Leftrightarrow H = Ha$.	**05**
	b)	Give an example of a non-commutative group whose all subgroups are normal subgroups.	**05**
	c)	Let $\phi: (G,*) \to (G^{'},*^{'})$ be an isomorphism. If N is a normal subgroup of G then $\phi(N)$ is a normal subgroup of G'.	**04**
Q-8		**Attempt all questions**	**(14)**
	a)	State and prove Cayley's Theorem.	**07**
	b)	Let $(G,*), (G^{'},*^{'})\ and\ (G",*")$ be a groups. If $(G,*) \cong (G^{'},*^{'})$ and $(G^{'},*^{'}) \cong (G",*")$ then prove that $(G,*) \cong (G",*")$.	**07**